J. Fluid Mech. (1977), *vol.* 83, *part* 2, *pp.* 209–224
Printed in Great Britain

Small perturbation expansions in unsteady aerofoil theory

By J. M. R. GRAHAM AND I. KULLAR
Department of Aeronautics, Imperial College, London

(Received 17 February 1977)

Perturbation expansions are derived to second order in a wavenumber parameter for the unsteady lift induced on an aerofoil by disturbances convected past it at subsonic speeds. The results are used to discuss other approximate methods which have been used to predict the unsteady forces and noise generated by an aerofoil in turbulent flow.

1. Introduction

The effects of small convected disturbances in the flow past an aerofoil are usually treated by Fourier analysis, since the problem is linear. The simplest Fourier component to study is a convected two-dimensional sinusoidal gust of vertical velocity. The effects of such a disturbance whose wave fronts are parallel to the leading edge of the aerofoil were first calculated by Sears (1941) for incompressible flow. This type of gust is often referred to as the Sears gust. The solution given by Sears was later generalized by Kemp (1952) to cases when the gust was convected at speeds different to the free-stream speed. Many other analyses of similar basic incompressible two-dimensional types of problem exist and the majority lead conveniently to analytic expressions for the lift and other force coefficients in terms of well tabulated functions.

Unfortunately however, no similar analytic solutions (in terms of a finite number of known functions) exist for those cases in which either the gust wave fronts are oblique to the aerofoil or the flow is compressible and subsonic. Accurate numerical results have been calculated for these cases by Graham (1970*a*, *b*) and by Adamczyk (1971). Results such as these are required for the general study of an aerofoil in a subsonic turbulent flow, but the numerical results are inconvenient because they require fairly lengthy computation followed by further integrations to obtain the response to turbulence and also because they do not explicitly reveal the importance of the various parameters involved. As a consequence a number of approximate solutions have been derived and used to predict the effects of turbulence on the unsteady forces and noise radiation.

In the case of an oblique gust in incompressible flow (i.e. a vertical velocity disturbance having both a non-zero chordwise and a non-zero spanwise wavenumber with respect to axes aligned with the aerofoil), Filotas (1969) and Mugridge (1971) have both given approximate solutions. More recently, Amiet (1976*a*) has adapted his compressible solution, discussed below, to this case by making use of similarity rules (Graham 1970*b*) which relate all compressible and/or oblique gusts to one problem. Filotas obtained his solution by assuming an empirical pressure distribution over the aerofoil, correct at zero and very high values of the spanwise wavenumber, and using this to obtain an approximate solution of the integral equation for the loading. In contrast, Mugridge's solution was obtained by approximating the integral equation according to the ideas of lifting-line theory, i.e. by neglecting chordwise vorticity on the aerofoil and using local two-dimensional solutions at each cross-section of the aerofoil.

The other problem, which has received more attention, is the extension of Sears' solution to compressible subsonic flow. This problem has been expressed in the form of an integral equation for the pressure distribution by Possio (1938). Miles (1950*a*) gave a quasi-steady solution to Possio's integral equation for small values of the reduced frequency parameter k_1 when the unsteady flow was caused by oscillations of the aerofoil rather than by a gust. His solution neglects terms of order k_1^2 and above but does include the effects of non-zero Mach number M not only through the usual $1/\beta\,[=1/(1-M^2)^{\frac{1}{2}}]$ Prandtl–Glauert amplitude correction, but also through a phase correction involving a function of M. This recognizes the essential point that compressibility introduces phase shifts, particularly in signals coming from the vortex wake. Reissner (1951) transformed the linear differential (convected wave) equation for the potential ϕ into a simpler form:

$$\phi_{xx}+\phi_{yy}+(k_1 M/\beta^2)^2\,\phi = 0,$$

where x and y are the streamwise and spanwise co-ordinates. The transformation introduces a factor of $\exp\{ik_1 M^2 x/\beta^2\}$ into the potential and hence the pressure distribution, but leaves the form of the boundary conditions unchanged except for the $1/\beta$ Prandtl–Glauert factor and a raising of the reduced frequency k_1 to k_1/β^2. Reissner went on to solve the equation by separation of variables, giving a solution for the problem of an oscillating wing in terms of infinite series of Matthieu functions.

It is apparent from the form of the above equation and its associated conditions that the problem involves two parameters:

$$\nu = k_1 M/\beta^2, \quad \lambda = k_1/\beta^2.$$

Osborne (1973) analysed the problem in this way and argued that a solution for small ν could be obtained to order ν by neglecting the $\nu^2\phi$ term in the differential equation and solving the remaining equation as an equivalent incompressible problem. Osborne's result for the lift coefficient on an aerofoil in a Sears-type gust is

$$C_L(k_1, M) = \beta^{-1}C_{L0}(\lambda)\,(J_0(M^2\lambda)-iJ_1(M^2\lambda)),$$

where C_{L0} is the incompressible lift coefficient given by Sears.

This method of solution implies a division of the flow field into an inner 'incompressible' region and an outer acoustic region. It has been criticized by Amiet (1976*b*), who based his argument on reasons given previously by Miles (1950*b*) that the inner incompressible flow field of the lifting aerofoil includes a vortex wake extending infinitely far downstream. In an earlier paper Amiet (1974) applies a correction to Osborne's solution based on the Miles (1950*a*) low frequency expansion mentioned above. This correction introduces a phase change dependent on the Mach number and improves the overall agreement with numerical results.

Kemp & Homicz (1976) have shown by an expansion of the Possio integral equation to first order that Amiet's solution is correct to this order in the frequency parameter λ. This result also becomes apparent from our analysis, which shows agreement to order $(\lambda\log\lambda)^2$ between this solution and a series solution derived for small reduced frequency k_1 and fixed Mach number. In fact because of its form Amiet's solution is quite accurate for all reduced frequencies at Mach numbers below 0·8. However transformation of his theory to the cases of incompressible oblique gusts, required for representation of turbulence, shows larger deviations from the accurate numerical results. (In the extreme case $k_1 = 0$, the error is about 50 % at a spanwise reduced frequency $k_2 = 1{\cdot}0$; see Amiet 1976*a*.)

A reason for this difference in accuracy is that two distinct types of perturbation expansion can be derived. The first, corresponding to the above cases, is an expansion for small λ for fixed finite values of ν/λ, i.e. an expansion for small values of the frequency k_1 at fixed Mach number in compressible two-dimensional flow and at a fixed angle between the gust wave front and the leading edge in incompressible oblique flow. The second type of expansion, which is the main subject of the present paper, is an expansion in powers of ν with λ held fixed, i.e. a small Mach number or small spanwise wavenumber perturbation from the incompressible Sears solution. This expansion seems to be more applicable than the first for incompressible oblique flows but more limited for compressible two-dimensional flows. The second type of expansion is also different from the first in that no first-order terms occur at all for the effects of compressibility or spanwise variability at fixed non-zero frequency. Therefore theories such as Mugridge's for incompressible oblique flows which implicitly assume the existence of a first-order term must also be considered as corresponding to the first type (that is, in principle, for small k_1 with k_2/k_1 fixed).

The method which we describe is an expansion of the integral equation for a function related to the unsteady pressure distribution as a series in terms of the small parameter ν. This is a similar approach to that of Kemp & Homicz (1976) taken to one order higher. The parameter ν is generally a combination of the modified spanwise wavenumber k_2/β^2 and the unsteady compressibility parameter $k_1 M/\beta^2$. The simplified integral equations of each order are then solved analytically in sequence, each equation being dependent on those of lower order. This process is equivalent to a matched asymptotic expansion, replacing the matching by the successive dependence of the integral equations. The similarity rules mentioned above enable the same method to be applied to all compressible oblique gusts with the parameter ν taking either small real or small imaginary values.

Since logarithmic terms are present in the perturbation series we adopt the usual convention of grouping these with all other terms containing the small parameter to the same power. Therefore, for example, ν^2 and $\nu^2 \log \nu$ are both considered to be of second order.

The series derived in this way are compared with the accurate numerical results to establish their range of validity.

We assume throughout that the aerofoil is thin, uncambered and at zero angle of attack, and that the gust is a sinusoidal distribution of vertical velocity convected at the free-stream speed. The effects of relaxing many of these conditions in the basic Sears-type problem have been studied by a number of authors. In principle all such cases which give analytical solutions can be extended to oblique compressible flows by the methods outlined below.

2. Expansion of the integral equation

We follow the notation of Sears throughout and take x, y and z axes in the streamwise, spanwise and normal directions respectively with origin at the midchord of the aerofoil. All velocities are non-dimensionalized by taking the mean free-stream speed as the unit of velocity and all lengths by the semichord of the aerofoil.

A general triple-wavenumber ($\mathbf{k}$) vertical gust can be written as

$$w(x,y,z,t) = \overline{w}(\mathbf{k})\exp\{i(\omega t - k_1 x - k_2 y - k_3 z)\} = \hat{w}(y,t)\exp(-ik_1 x - ik_3 z), \quad \text{say}.$$

$\overline{w}(\mathbf{k})$ is a Fourier coefficient of a turbulent flow field or any other convected three-dimensional disturbance. It is convenient here to use the quantity $\hat{w}$ to avoid continuous repetition of the factor $\exp\{i(\omega t - k_2 y)\}$, which is common throughout the analysis.

The gust is assumed to be convected with the free stream, so that $\omega = k_1$. Then, following Graham (1970*b*), the coefficient of loading $C_{\Delta p}(x,y,t)$ on the aerofoil is governed by the following integral equation and associated relations:

$$\int_{-1}^{1} \nu F(x_0) K_1(\nu(x_0 - x))\,dx_0 = -\pi\hat{w}\exp(-i\lambda x) + \nu^2 \int_{-1}^{1}\int_{-1}^{x_0} f(x_1)\,dx_1\,dx_0$$
$$\times K_0(\nu(1-x)) - i\lambda \int_{-1}^{1} f(x_0)\,dx_0 \left\{\frac{\nu^2}{\lambda^2}K_0(\nu(1-x)) - \left(1+\frac{\nu^2}{\lambda^2}\right)\int_0^\infty \nu\exp(-i\lambda x_0)\right.$$
$$\left.\times K_1(\nu(1+x_0-x))\,dx_0\right\}, \tag{1}$$

where
$$F(x) = f(x) - \nu^2\int_{-1}^{x}\int_{-1}^{x_0} f(x_1)\,dx_1\,dx_0,$$

and
$$C_{\Delta p}(x,y,t) = \frac{2}{\beta}\exp\{ik_1 M^2 x/\beta^2\}\left(f(x) - i\lambda\int_{-1}^{x} f(x_0)\,dx_0\right). \tag{2}$$

The K_n are modified Bessel functions with their analytic continuation for complex arguments and
$$\nu = (k_2^2/\beta^2 - k_1^2 M^2/\beta^4)^{\frac{1}{2}}, \quad \lambda = k_1/\beta^2. \tag{3}$$

The third wavenumber k_3 of the gust does not enter the problem unless distortion of the vorticity in the gust is taken into account (see, for example, McKeough 1976), since the aerofoil lies in the plane $z = 0$.

The main difficulty in the way of an analytical solution of (1) is the form of the integrand on the left-hand side. In the special cases when $\nu = 0$ this integral takes the simpler form
$$\int_{-1}^{1}\frac{F(x_0)}{x_0 - x}\,dx_0.$$

This is the integral of two-dimensional thin-aerofoil theory, which can be inverted by a standard procedure. When both k_2 and M are zero, ν is zero also and the solution of (1) for the lift coefficient
$$C_L = \frac{1}{2}\int_{-1}^{1} C_{\Delta p}(x_0)\,dx_0$$

is
$$C_L(k_1, k_2 = 0, M = 0) = 2\pi\hat{w}\frac{J_0(k_1)H_1^{(2)}(k_1) - J_1(k_1)H_0^{(2)}(k_1)}{H_1^{(2)}(k_1) + iH_0^{(2)}(k_1)} = 2\pi\hat{w}S(k_1), \quad \text{say}. \tag{4}$$

This is the result given by Sears (1941), which we take as our basic solution C_{L0}.

There are also non-zero values of k_2 and M for which $\nu = 0$ but for which $C_L \neq C_{L0}$ because of the factor $\exp\{ik_1 M^2 x/\beta^2\}$ in (2). When ν is non-zero it takes real or imaginary values according to whether $k_1 M/k_2\beta \gtrless 1$. This corresponds physically to whether, relative to a frame of reference moving spanwise, so that the gust appears stationary, the velocity of the free stream is subsonic (ν real) or supersonic (ν imaginary). ν is assumed in either case to be the positive square root.

To illustrate the method of expansion we first consider incompressible flow, so that

$$M = 0, \quad \nu = k_2 = \nu_2, \quad \text{say}, \quad \lambda = k_1.$$

Then expanding (1) for small real ν_2, but with $\lambda = O(1)$ so that ν_2/λ is also small, gives

$$\begin{aligned}
&\int_{-1}^{1} F(x_0)\{1/(x_0-x) + \tfrac{1}{2}(x_0-x)\,\nu_2^2\log\nu_2 + \tfrac{1}{2}(x_0-x)\,(\gamma - \log 2 - \tfrac{1}{2} + \log(x-x_0))\,\nu_2^2\}\,dx_0 \\
&\quad = -\pi\hat{w}\exp(-i\lambda x) + \{-\nu_2^2\log[\nu_2(1-x)] + \nu_2^2(\log 2 - \gamma)\}\int_{-1}^{1}\int_{-1}^{x_0} f(x_1)\,dx_1\,dx_0 \\
&\qquad - i\lambda\int_{-1}^{1} f(x_0)\,dx_0\{i(1+\nu_2^2/2\lambda^2)\exp[i\lambda(1-x)]\,(\tfrac{1}{2}\pi - i\log 2 + i\log(\nu_2/\lambda) - i\nu_2^2/4\lambda^2) \\
&\qquad + \log[\nu_2(1-x)] - \log(2-\gamma) - \nu_2^2/4(\log 2 - \gamma + 1)\,(1-x)^2 + \nu_2^2/4(1-x)^2\log[\nu_2(1-x)] \\
&\qquad + (1+\nu_2^2/\lambda^2)\exp[i\lambda(1-x)]\,[-\exp[-i\lambda(1-x)]\log[\nu_2(1-x)] + \log(\nu_2/\lambda) - \gamma \\
&\qquad + \mathrm{Ci}\,[\lambda(1-x)] - i\,\mathrm{Si}\,[\lambda(1-x)] - ((\log 2 - \gamma)\,(1 - \exp[-i\lambda(1-x)]) \\
&\qquad - \nu_2^2/4\lambda^2(\log 2 - \gamma + 1 - \log\nu_2)\,[(-\lambda^2(1-x)^2 + 2i\lambda(1-x) + 2)\exp[-i\lambda(1-x)] - 2] \\
&\qquad + \nu_2^2/4\lambda^2(\exp[-i\lambda(1-x)]\,[-\lambda^2(1-x)^2\log(1-x) \\
&\qquad + 2i\lambda(1-x)\log(1-x) + i\lambda\,(1-x) + 3 \\
&\qquad + 2\log(1-x)] + 2\log\lambda - 3 - 2(\mathrm{Ci}\,[\lambda(1-x)] - i\,\mathrm{Si}\,[\lambda(1-x)] - \gamma))]\} + O(\nu_2^3).
\end{aligned} \tag{5}$$

γ is Euler's constant and Ci and Si are the cosine and sine integrals. Both sides of (5) contain only terms of order 1, $\nu_2^2\log\nu_2$, ν_2^2 and higher. Therefore any $O(\nu_2)$ term which might occur in the expansion is the solution of an integral equation containing neither the upwash $\hat{w}$ nor a non-zero correction to it from a lower-order term, and must be itself zero. We therefore expand $F(x)$, $f(x)$ and C_L as

$$\begin{aligned}
F(x) &= F_0(x) + F_1(x)\,\nu_2^2\log\nu_2 + F_2(x)\,\nu_2^2 + \dots, \\
f(x) &= f_0(x) + f_1(x)\,\nu_2^2\log\nu_2 + f_2(x)\,\nu_2^2 + \dots, \\
C_L &= C_{L0} + C_{L1}\,\nu_2^2\log\nu_2 + C_{L2}\,\nu_2^2 + \dots.
\end{aligned}$$

Substituting these into (5) and equating terms of like order, we obtain a series of integral equations of the form

$$\int_{-1}^{1}\frac{F_n(x_0)}{x_0 - x}\,dx_0 = G_n(\hat{w}\delta_{n0}, f_{n-m}, F_{n-m}), \quad n \geqslant m > 0,$$

where δ_{n0} is the Kronecker delta.

The F_{n-m} terms, which are always of lower order, since $m \geqslant 1$, if they occur at all, come from parts of the left-hand-side integrand of (5) which have been taken over to the other side of the equation. These and similar f_{n-m} terms can be assumed to have already been calculated from solutions of lower-order equations and can be considered as an upwash correction. The equations for each order were solved by the integral-equation method described by Bisplinghoff, Ashley & Halfman (1955) for the Sears problem. The first-order equation is identical with the Sears problem.

After manipulation and integration† we obtain the following series for the lift coefficient:

$$C_L(k_1, k_2) = 2\pi\hat{w}\left\{S(\lambda) + \left[\frac{1}{4} + \frac{1}{2i\lambda}C(\lambda)\right] S(\lambda)\,\nu_2^2 \log \nu_2 + [A(\lambda)\,S(\lambda) + B(\lambda)]\,\nu_2^2 + O(\nu_2^3)]\right\}. \tag{6}$$

In this equation

$$A(\lambda) = \tfrac{1}{4}(\gamma - 2\log 2) + (\gamma - 2\log 2 - \tfrac{3}{2})\,C(\lambda)/2i\lambda,$$

$$B(\lambda) = \left(\frac{3}{8} - \frac{i}{4\lambda}\right) J_0(\lambda)\,C(\lambda) + \frac{i}{4\lambda} J_0(\lambda)$$
$$-\left(\frac{3i}{8} - \frac{1}{2\lambda} - \frac{i}{2\lambda^2}\right) J_1(\lambda)\,C(\lambda) - \left(\frac{i}{8} + \frac{i}{2\lambda^2}\right) J_1(\lambda) - \tfrac{1}{2}(J_0(\lambda) - iJ_1(\lambda))\,C(\lambda)$$
$$+ \frac{J_0(\lambda) - iJ_1(\lambda)}{2i\lambda(H_1^{(2)}(\lambda) + iH_0^{(2)}(\lambda))}\left(\frac{i}{2}H_0^{(2)}(\lambda)\,[1 - C(\lambda)] + H_1^{(2)}(\lambda)\left[\frac{1}{i\lambda} + C(\lambda)\left(1 + \frac{i}{\lambda}\right)\right]\right),$$

$C(\lambda)$ = Theodorsen's function $= H_1^{(2)}(\lambda)/[H_1^{(2)}(\lambda) + iH_0^{(2)}(\lambda)]$ and $S(\lambda)$ is Sears' function, given in (4).

The absence of an $O(\nu_2)$ term in this perturbation series contrasts with the high aspect ratio ($\mathscr{A}$) perturbation series for a finite-span wing in a two-dimensional unsteady flow derived by James (1975) and Van Holten (1976). They both find that, provided k_1 is $O(1)$, $C_L(k_1, \mathscr{A}) = C_{L0}(k_1) + \mathscr{A}^{-1}C_{L1}(k_1) + \ldots$. The perturbation parameter $\mathscr{A}^{-1}$ is analogous to the spanwise wavenumber parameter $\nu_2(= k_2)$. In the steady case $k_1 = 0$, the oblique gust series which we derive below does contain an $O(k_2)$ term which is analogous to the $\mathscr{A}^{-1}$ term of steady lifting-line theory. The centre-section lift coefficient on a rectangular wing which is suitably twisted to have an elliptic spanwise lift distribution is

$$C_L(y = 0) = 2\pi\alpha_0\{1 - \tfrac{1}{2}\pi\mathscr{A}^{-1}\ldots +\},$$

where α_0 is the angle of incidence at $y = 0$. The lift on an infinite-span, sinusoidally twisted wing of spanwise wavelength $2\pi/k_2$, which is equivalent to an oblique gust with $k_1 = 0$, is

$$C_L = 2\pi\alpha_0\{1 - \tfrac{1}{2}\pi k_2 \ldots +\}.$$

Since all lengths are non-dimensionalized by the semichord, the wavelength of the twisted wing/oblique gust must be equal to π times the span of the finite wing for the two correction terms to be equal. But when the oncoming flow is unsteady, or the wing oscillates, the two cases of an oblique gust and a finite wing are not comparable unless k_1 is large ($O(\mathscr{A})$), when James and Van Holten show that the first perturbation term is $O(\mathscr{A}^{-2}\log \mathscr{A})$.

Figure 1 shows the region in the k_1, k_2 plane within which the perturbation series (6) for the lift coefficient differs from the accurate numerical value by less than about 10 % of its magnitude. Of course the series fails to give accurate results for sufficiently large values of k_2 but it also fails if k_1 is too small. This is because the expansion does not simply involve small ν_2 but also requires the parameter ν_2/λ ($= k_2/k_1$) to be small. For example (6) gives an infinite result for the steady case $k_1 = 0$, $k_2 \neq 0$.

Because of the occurrence of ν_2/λ in (1) we might expect that different series would be required for the cases

$$\nu_2 \to 0, \quad \lambda = O(\nu_2),$$
$$v_2 \to 0, \quad \lambda = O(1), \text{ which is (6)},$$

† The details are available from the authors as an Imperial College, Aeronautics Department Internal Report.

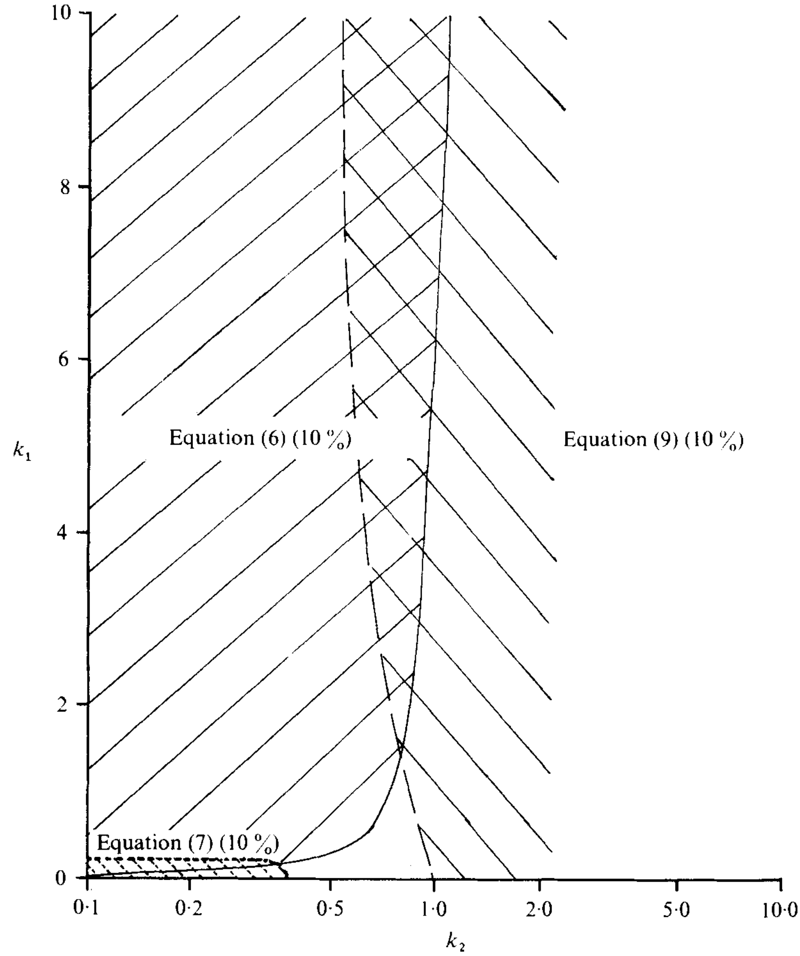

FIGURE 1. Ranges of application of incompressible oblique gust theories.

and $$\nu_2 \to 0, \quad \lambda = O(1/\nu_2).$$

In fact series (6) is valid for the last case as well as the second and we require only one additional series, for $\nu_2 \to 0$, $\lambda = O(\nu_2)$. This case corresponds to the expansion for small λ with ν/λ fixed, discussed in the introduction.

Returning to (1) and writing $k_1 = \lambda = \alpha\nu_2$, where $\alpha = O(1)$, gives

$$\int_{-1}^{1} F(x_0)\left\{\frac{1}{x_0-x}+\left(\frac{x_0-x}{2}\right)\nu_2^2\log\nu_2+\left(\frac{x_0-x}{2}\right)(\gamma-\log 2-\tfrac{1}{2}+\log(x_0-x))\,\nu_2^2\right\}dx_0$$
$$= -\pi\hat{w}\left(1-i\alpha\nu_2 x-\frac{\alpha^2\nu_2^2x^2}{2}\right)+\left(\int_{-1}^{1}\int_{-1}^{x_0} f(x_1)\,dx_1\,dx_0\right)\{(\log 2-\gamma)\,\nu_2^2-\nu_2^2\log\nu_2$$
$$-\nu_2^2\log(1-x)\}-\left(\int_{-1}^{1} f(x_0)\,dx_0\right)\{-\eta\delta\nu_2+i\alpha\nu_2[\log(1-x)-i\log 2+i\gamma]+i\alpha\nu_2\log\nu_2$$
$$-i\alpha\eta\delta(1-x)\,\nu_2^2-\eta^2(1-x)\,\nu_2^2\log\nu_2-\eta^2\nu^2(1-x)\log(1-x)$$
$$+(\log 2-\gamma+1)\,\eta^2\nu_2^2(1-x)\}+O(\nu_2^3),$$

where $\eta = (1+\alpha^2)^{\frac{1}{2}}$ and $\delta = \frac{1}{2}\pi - i\log(\alpha+\eta)$. The right-hand side of this equation contains terms of order ν_2 and $\nu_2 \log \nu_2$ as well as the terms of other orders in ν_2 which previously appeared. Therefore, arguing as before, we expand $F(x)$ as

$$F(x) = F_0(x) + \nu_2 \log \nu_2 F_1(x) + \nu_2 F_2(x) + \nu_2^2(\log \nu_2)^2 F_3(x) + \nu_2^2 \log \nu_2 F_4(x) + \nu_2^2 F_5(x) + O(\nu_2^3),$$

with similar expansions for $f(x)$ and C_L.

Solving the integral equations of each order in turn, we obtain for this case ($k_1/k_2 < 1$)

$$\begin{aligned} C_L(k_1,k_2) = 2\pi\hat{w}\{&1 + i\lambda\log\nu_2 - \eta\delta\nu_2 - i(2\log 2-\gamma)\lambda - \lambda^2(\log\nu_2)^2 \\ &+ \tfrac{1}{2}\eta^2\nu_2^2\log\nu_2 - 2i\eta\delta\lambda\nu_2\log\nu_2 + 2(2\log 2-\gamma)\lambda^2\log\nu_2 + (\tfrac{1}{2}\gamma - \tfrac{1}{4} - \log 2 + \delta^2)\eta^2\nu_2^2 \\ &+ i\eta\delta(\tfrac{1}{2} + 4\log 2 - 2\gamma)\lambda\nu_2 - (2\log 2-\gamma)^2\lambda^2 + O(\nu_2^3)\}, \end{aligned} \quad (7)$$

where $\lambda = k_1$, $\nu_2 = k_2$, $\eta = (1+k_1^2/k_2^2)^{\frac{1}{2}}$ and $\delta = \frac{1}{2}\pi - i\log[k_1/k_2 + (1+k_1^2/k_2^2)^{\frac{1}{2}}]$. Series (7) is the correct expansion when both k_1 and k_2 are small, including the two cases k_1 small, $k_2 = 0$ and $k_1 = 0$, k_2 small. The former case gives

$$\begin{aligned} C_L(k_1\to 0, 0) = 2\pi\hat{w}\{&1 + ik_1\log k_1 - (\tfrac{1}{2}\pi + i\log 2 - i\gamma)k_1 \\ &- k_1^2(\log k_1)^2 + (\tfrac{1}{2} - i\pi + 2\log 2 - 2\gamma)k_1^2\log k_1 \\ &+ [\tfrac{1}{2}\gamma - \tfrac{1}{4} - \tfrac{1}{2}\log 2 + \tfrac{1}{4}i\pi + \tfrac{1}{4}\pi^2 + i\pi\log 2 \\ &- (\log 2)^2 - i\pi\gamma + 2\gamma\log 2 - \gamma^2]k_1^2\}. \end{aligned} \quad (8)$$

Equation (8) is the same as the small-k_1 expansion of Sears' result. It has however a limited range of validity, as Amiet (1974) has previously observed, mainly because of the limited range of accuracy of the trinomial $(1 - ik_1x - \frac{1}{2}k_1^2x^2)$ representation of $\exp(-ik_1x)$. The second of the two cases gives

$$C_L(0, k_2\to 0) = 2\pi\hat{w}\{1 - \tfrac{1}{2}\pi k_2 + \tfrac{1}{2}k_2^2\log k_2 + (\tfrac{1}{2}\gamma - \tfrac{1}{4} - \log 2 + \tfrac{1}{4}\pi^2)k_2^2\},$$

which is the small perturbation expansion for a sinusoidally twisted wing of infinite span in steady flow, discussed above.

The range of validity of expansion (7), moderately small values of k_2 but very small values of k_1, is indicated in figure 1. Expansions (6) and (7) are sufficient to cover the whole range of small k_2. When k_2 is large, Filotas' (1969) approximate solution provides an asymptotically correct result:

$$C_L(k_1,k_2) = 2^{\frac{3}{2}}\hat{w}\exp(ik_1)\frac{(k+k_2)^{\frac{1}{2}} - i(k-k_2)^{\frac{1}{2}}}{(k_1^2 + 4k_2k^2)^{\frac{1}{2}}}, \quad (9)$$

where $k = (k_1^2 + k_2^2)^{\frac{1}{2}}$. The region of accuracy of (9) is also marked in figure 1, and it is apparent from this that these expansions, between them, cover most of the k_1, k_2 plane to an accuracy of 10 %.

3. Two-dimensional compressible unsteady flow

We consider next the simplest case when the free-stream Mach number is not negligible. This is the case of a two-dimensional compressible unsteady flow ($k_2 = 0$). The small parameter ν defined in (3) is now imaginary and the integral equation (2) is equivalent to Possio's integral equation since

$$K_n(iz) = -\tfrac{1}{2}\pi i\, e^{-\frac{1}{2}n\pi i}H_n^{(2)}(z).$$

By repeating the incompressible flow analysis with $\nu = ik_1 M/\beta^2 = i\nu_1$, say, and $\lambda = k_1/\beta^2$ and taking account of the additional factor

$$\exp\{ik_1 M^2 x/\beta^2\} = \exp\{i\nu_1^2 x/\lambda\},$$

which is expanded for small argument, we obtain the compressible flow series:

$$C_L(k_1, M) = \frac{2\pi\hat{w}}{\beta}\left\{S(\lambda) - \left[\frac{1}{4} + \frac{C(\lambda)}{2i\lambda}\right] S(\lambda)\,\nu_1^2 \log \nu_1 - \left\{\left[\frac{i\pi}{2}\left(\frac{1}{4} + \frac{C(\lambda)}{2i\lambda}\right) + A(\lambda) - \frac{1}{2i\lambda}\right] S(\lambda) + B(\lambda)\right\} \nu_1^2 + O(\nu_1^3)\right\}, \quad (10)$$

where $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $S(\lambda)$ are as defined for (6). This result fails, as the incompressible one does, when $\lambda = O(\nu_1)$.

Since $\nu_1/\lambda = M$, λ is always greater than ν_1 for subsonic flows. But in contrast to (6), (10) gives poor accuracy for the lift coefficient when ν_1/λ exceeds fairly small values of the Mach number. The expansion corresponding to (7) should be appropriate for this region when the frequency (and therefore λ) is sufficiently small. It can be similarly derived, again taking account of the factor $\exp(iM\nu_1 x)$, to give

$$C_L(k_1, M; \nu_1 \to 0, M = O(1)) = (2\pi\hat{w}/\beta)\{1 + i\lambda\log\lambda + i\lambda[f(M) + C_0 - \tfrac{1}{2}M^2] - \lambda^2(\log\lambda)^2 - \lambda^2\log\lambda[2f(M) + 2C_0 - \tfrac{1}{2}] - \lambda^2[(f(M)+C_0)^2 - \tfrac{1}{2}(1+M^2)(f(M)+\tfrac{1}{2}M^2) + \tfrac{1}{2}M^2\log(\tfrac{1}{2}M) + \tfrac{1}{4} - \tfrac{1}{2}C_0] + O(\lambda^3)\}, \quad (11)$$

where
$$f(M) = (1-\beta)\log M + \beta\log(1+\beta) - \log 2$$

and
$$C_0 = \tfrac{1}{2}i\pi - \log 2 + \gamma.$$

The asymptotic compressible result analogous to the high wavenumber result of Filotas for incompressible flow [equation (9)] is the transonic ($M = 1$) result given in Graham (1970*b*). Either of these two formulae can also be deduced from the other by applying the similarity rules. The region of the k_1, M plane of accuracy of (10) and (11) is shown in figure 2.

4. Oblique compressible gusts

We adopt the same procedure for the general oblique, compressible case ($k_2 \neq 0$, $M \neq 0$) with

$$\nu = (k_2^2/\beta^2 - k_1^2 M^2/\beta^4)^{\frac{1}{2}}$$

real or imaginary according to the size of $\theta = k_1 M/k_2\beta$. However, the factor

$$\exp\{ik_1 M^2 x\beta^2\}$$

now introduces an additional complication because $k_1 M^2/\beta^2$ can no longer be expressed as a simple combination of λ and ν. It is therefore convenient to consider the expansion in terms of the two small parameters

$$\nu_1 = k_1 M/\beta^2, \quad \nu_2 = k_2/\beta.$$

If the expansion is derived for small ν_1 and ν_2 then the general parameter ν, whose absolute value is never greater than the larger of these, is also small. The factor $\exp\{ik_1 M^2 x/\beta^2\}$ can be written as before as $\exp\{i\nu_1^2 x/\lambda\}$ and expanded for small

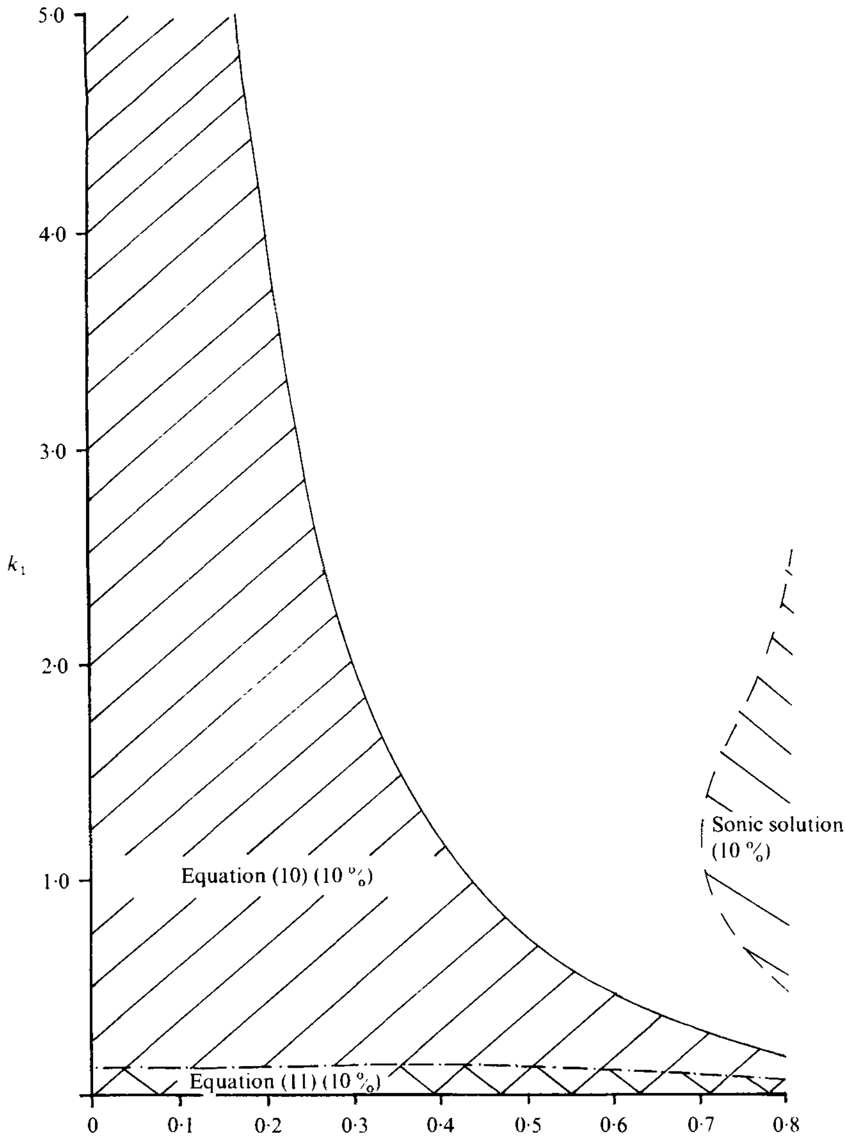

FIGURE 2. Ranges of application of compressible two-dimensional gust theories.

argument. The result for a compressible oblique subsonic gust for which $k_1 M/\beta^2 = \nu_1 \ll 1$, $k_2/\beta = \nu_2 \ll 1$ and $\lambda = k_1/\beta^2 = O(1)$ is

$$C_L(k_1, k_2, M) = \frac{2\pi\hat{w}}{\beta}\left[S(\lambda) - \frac{1}{2}\left(\frac{1}{4}+\frac{C(\lambda)}{2i\lambda}\right) S(\lambda)\,\nu_1^2 \log(\nu_1^2-\nu_2^2)\right.$$
$$-\left\{\left[\frac{i\pi}{2}\left(\frac{1}{4}+\frac{C(\lambda)}{2i\lambda}\right)+A(\lambda)+\frac{1}{2i\lambda}\right]S(\lambda)+B(\lambda)\right\}\nu_1^2$$
$$\left.+\frac{1}{2}\left(\frac{1}{4}+\frac{C(\lambda)}{2i\lambda}\right)S(\lambda)\,\nu_2^2\log(\nu_2^2-\nu_1^2) + (A(\lambda)S(\lambda)+B(\lambda))\,\nu_2^2 + O(\nu_1^3, \nu_2^3)\right], \quad (12)$$

where $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $S(\lambda)$ are as defined in (6).

This result contains both the results (6) and (10) given previously. In the special cases when $\nu_1 = \nu_2$ ($\theta = 1$), the second-order logarithmic terms cancel with part of the ν_1^2 term, and so do the $A(\lambda)$ and $B(\lambda)$ terms. The result is simply

$$C_L(\theta = 1) = \frac{2\pi\hat{w}}{\beta}\left(1 - \frac{\nu_1^2}{2i\lambda}\right)S(\lambda) + O(\nu_1^3).$$

As in the incompressible case the expansion (11) fails when $\lambda = O(\nu_2)$. In this case the expansion analogous to (7) applies:

$$\begin{aligned} C_L(k_1, k_2, M) = (2\pi\hat{w}/\beta)\{1 &+ \tfrac{1}{2}i\lambda\log(\nu_2^2 - \nu_1^2) - \eta\delta(\nu_2^2 - \nu_1^2) - i\lambda(2\log 2 - \gamma) + \nu_1^2/2i\lambda \\ &- \tfrac{1}{4}\lambda^2[\log(\nu_2^2 - \nu_1^2)]^2 + \tfrac{1}{4}\eta^2(\nu_2^2 - \nu_1^2)\log(\nu_2^2 - \nu_1^2) + \tfrac{1}{2}\nu_1^2\log\nu_1 \\ &- i\eta\delta\lambda(\nu_2^2 - \nu_1^2)^{\frac{1}{2}}\log(\nu_2^2 - \nu_1^2) + (2\log 2 - \gamma)\lambda^2\log(\nu_2^2 - \nu_1^2) \\ &+ (\tfrac{1}{2}\gamma - \tfrac{1}{4} - \log 2 + \delta^2)\eta^2(\nu_2^2 - \nu_1^2) + i\eta\delta(\tfrac{1}{2} + 4\log 2 - 2\gamma)\lambda(\nu_2^2 - \nu_1^2)^{\frac{1}{2}} \\ &- (2\log 2 - \gamma)^2\lambda^2 - \tfrac{1}{2}\eta(\nu_2 = 0)\delta(\nu_2 = 0)\lambda\nu_1 + (\tfrac{1}{4}i\pi - \log 2 + \tfrac{1}{2}\gamma)\nu_1^2 - \nu_1^4/4\lambda^2 + O(\nu_1^3)\}, \end{aligned} \tag{13}$$

where $$\eta = \left(1 + \frac{\lambda^2}{(\nu_2^2 - \nu_1^2)}\right)^{\frac{1}{2}}, \quad \delta = \frac{\pi}{2} - i\log\left(\frac{\lambda}{(\nu_2^2 - \nu_1^2)^{\frac{1}{2}}} + \eta\right).$$

The conditions for expansion (13) exclude the possibility that $\nu_1 = \nu_2$.

5. Comparison with other approximate formulae

Mugridge's (1971) formula based on lifting-line theory for unsteady incompressible flow is

$$C_L(k_1, k_2, M = 0) = 2\pi\hat{w}S(k_1)F(k_1, k_2).$$

The function F is a rational combination of Bessel and other functions for all k_1 and k_2. We can expand F for small k_2/k_1 to give

$$\begin{aligned} C_L = 2\pi\hat{w}\Bigg\{S(k_1) &- \frac{J_0(k_1) - iJ_1(k_1)}{\pi k_1^2(H_1^{(2)}(k_1) + iH_0^{(2)}(k_1))}S(k_1)k_2^2\log k_2 \\ &+ \left(\frac{i\pi}{2} + \log 2 + \log k_1\right)\frac{J_0(k_1) - iJ_1(k_1)}{\pi k_1^2(H_1^{(2)}(k_1) + iH_0^{(2)}(k_1))}S(k_1)k_2^2 + O(k_2^3)\Bigg\}. \end{aligned}$$

Comparing this equation with (5) shows that Mugridge's formula contains the correct orders of k_2 but not the correct coefficients. On the other hand the expansion of F for the quasi-steady case $k_1 = O(k_2)$ with both small agrees with (7) to first order in k_2. Failure of lifting-line theory in the general unsteady case ($k_1 = O(1)$) occurs because the downwash velocity correction due to the trailing vorticity is of the same higher order ($k_2^2\log k_2$, etc.) as the error incurred by applying locally two-dimensional analysis at each section of the aerofoil. For lifting-line theory to work, these error terms (typically $O(\mathscr{A}^{-2}\log\mathscr{A})$ etc.), which are neglected, must be of higher order than the downwash correction (typically $O(\mathscr{A}^{-1})$), which is incorporated.

Filotas' (1969) formula for the incompressible lift coefficient is

$$C_L = 2\hat{w}\frac{I_0(k_2) + I_1(k_2)}{J_0(k_2 + ik_1) - iJ_1(k_2 + ik_1)}\{k + \exp(ik_1)[k_2K_1(k_2) + ik_1K_0(k_2)] + \sigma\}^{-1}, \tag{14}$$

where the function σ involves double sums of repeated integrals of $K_0(k_2)$. As mentioned above, the asymptotic expression derived from (14) for large k is correct. But (14) is not correct for small k, where it becomes

$$C_L = 2\pi\hat{w}\{1 + ik_1 \log k + O(k)\}.$$

This is different from expansion (7) in all terms after the first. Likewise the expansion of (14) for small $k_2 = O(k_1)$ differs from (6) because Filotas' result only *approximates* $S(k_1)$ to order k_1 when $k_2 = 0$.

Amiet (1974, 1976*a*) also gives approximate solutions for oblique incompressible and compressible gusts, but as these are derived from his compressible result through the similarity rules it is sufficient to consider only the latter.

For the compressible two-dimensional gust Amiet (1974) has improved Osborne's (1973) solution by comparing the predicted pressure distribution with the earlier quasi-steady result of Miles (1950*a*). By analogy with this Amiet derives the following formula for the lift coefficient:

$$C_L(k_1, M) = (2\pi\hat{w}/\beta)\, S(\lambda)\, \{J_0(\nu_1^2/\lambda) - iJ_1(\nu_1^2/\lambda)\}\, e^{i\lambda f(M)}, \tag{15}$$

where $f(M) = (1-\beta)\log M + \beta \log(1+\beta) - \log 2$ and $\nu_1 = k_1 M/\beta^2$ and $\lambda = k_1/\beta^2$ as before. This expression gives the same amplitude for the lift as Osborne, shown in figure 3 for comparison with the perturbation series (10) and the numerical results. But the phase change brought about by the $\exp\{i\lambda f(M)\}$ factor considerably improves the accuracy of the individual real and imaginary parts of C_L, particularly at higher values of M; see figure 4. The reason for this can be seen if Amiet's expression (15) is expanded as a series in ν_1, assumed small. Then

$$C_L(k_1, M) = \frac{2\pi\hat{w}}{\beta} S(\lambda) \left\{1 - \frac{1}{2i\lambda}\nu_1^2 \log \nu_1 + \frac{1}{i\lambda}\left(\tfrac{1}{2}\log 2 + \tfrac{1}{2}\log\lambda + \tfrac{3}{4}\right)\nu_1^2 + O(\nu_1^3)\right\}. \tag{16}$$

This series does contain a compressibility phase-change term of order $\nu_1^2 \log \nu_1$, which was absent from Osborne's solution. Comparison with (10) shows that (16) contains some but not all of the terms which make up the coefficients of $\nu_1^2 \log \nu_1$ and ν_1^2 in (10) when this series is in turn expanded for small λ. Amiet's formula, like Mugridge's, does not therefore approach the $\nu_1 = 0$ Sears limit in the correct way unless k_1 also tends to zero. On the other hand it does provide an approximate solution to the problem which is never very inaccurate (figures 3 and 4) whereas the perturbation solution (10) rapidly diverges beyond $\nu_1 \sim 0{\cdot}5$.

Amiet's method of approximation is analogous to Mugridge's in that both approaches implicitly assume that $O(\nu)$ terms will occur in the expansion in relation to which the higher-order ν^2 terms in the differential equation can be ignored. Both methods thereafter solve the problem as far as possible without further approximation, so that parts of the correct higher-order terms occur in their solutions. But as is the case with the classical steady lifting-line theory the resulting expressions are moderately accurate, and in particular do not diverge, over the whole range $(0, \infty)$ of the small perturbation parameter ν.

If, however, (15) is expanded to second order for small k_1, with M held constant, we obtain the following:

$$\begin{aligned} C_L(k_1 M; k_1 \to 0, M = O(1)) = {} & (2\pi\hat{w}/\beta)\{1 + i\lambda \log\lambda + i\lambda[f(M) + C_0 - \tfrac{1}{2}M^2] \\ & - \lambda^2(\log\lambda)^2 - \lambda^2 \log\lambda[f(M) + 2C_0 - \tfrac{1}{2}(1+M^2)] - \lambda^2[C_0 f(M) - \tfrac{1}{2}C_0 M^2 + \tfrac{1}{2}f^2(M) \\ & - \tfrac{1}{2}M^2 f(M) + \tfrac{1}{4}M^4 + C_0^2 - \tfrac{1}{2}C_0 + \tfrac{1}{4}] + O(\lambda^3)\}. \end{aligned}$$

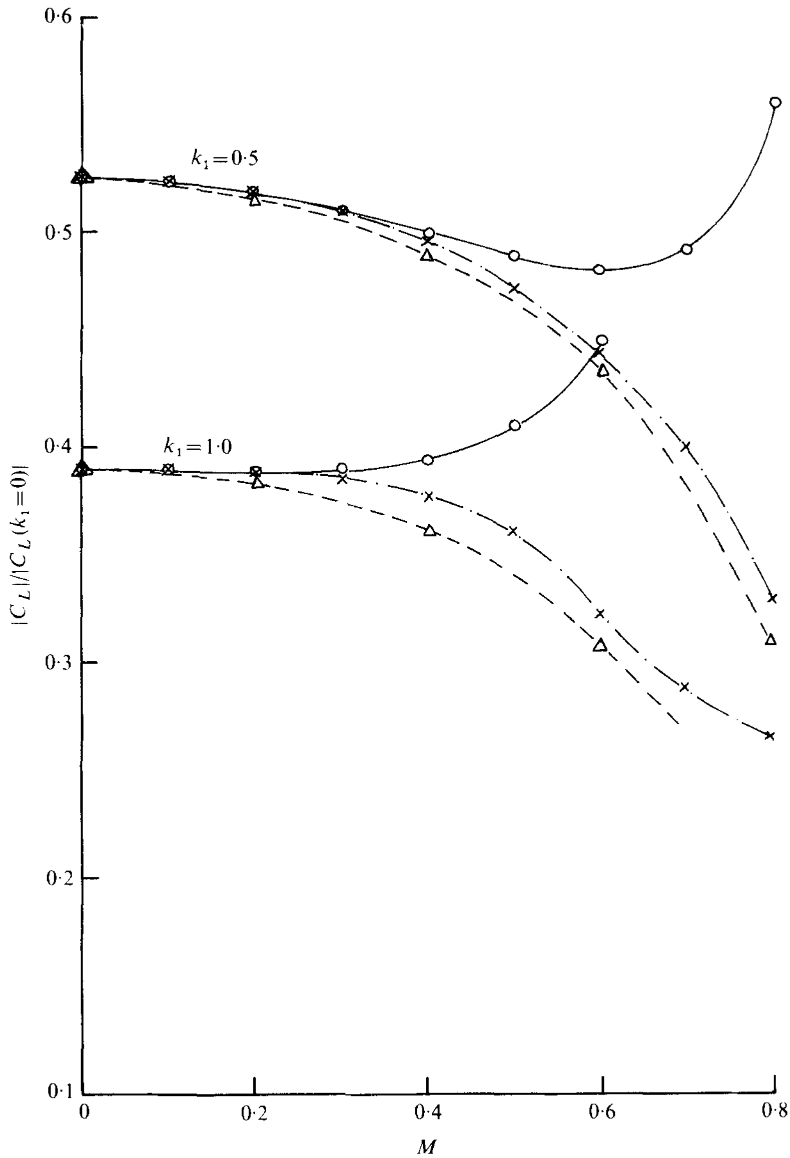

FIGURE 3. Variation of the lift-coefficient amplitudes with Mach number in compressible unsteady flow. —·– × –·—, numerical; —○—, equation (10); – –△– –, Amiet (1974), Osborne (1973).

This agrees with (11)† exactly up to order $k_1^2(\log k_1)^2$ (or $\nu_1(\log \nu_1)^2$). There is a difference in the $k_1^2 \log k_1$ and k_1^2 terms of

$$C_L(\text{eqn } 11) - C_L(\text{eqn } 15) = (2\pi\hat{w}/\beta)\{\lambda^2 \log \lambda(f(M) + \tfrac{1}{2}M^2) - \lambda^2[\tfrac{1}{2}f^2(M) + C_0 f(M) + \tfrac{1}{2}M^2\{\log(\tfrac{1}{2}M) - f(M) - C_0 - \tfrac{1}{2}\}] + O(\lambda^3)\}.$$

Kemp & Homicz (1976) have shown Amiet's result to be correct to first order in k_1, and the present analysis merely extends this, showing that the $k_1^2(\log k_1)^2$ term is also

† We are indebted to one of the referees for suggesting this comparison.

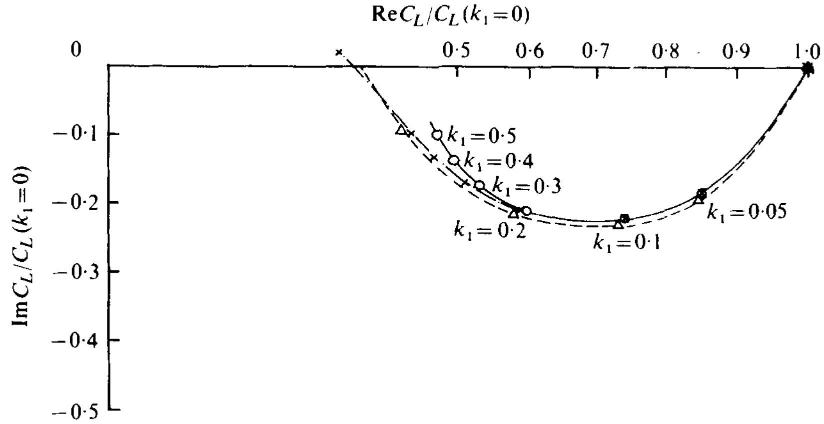

FIGURE 4. Variation of the lift coefficients with reduced frequency in compressible unsteady flow. $M = 0{\cdot}6$. —·—×—·—, numerical; —○—, equation (10); ‒‒△‒‒, Amiet (1974).

correct, when considered as an expansion for low frequencies at a fixed Mach number. The fact that the next two terms $k_1^2 \log k_1$ and k_1^2 are different reflects the removal of the $\nu_1^2 \phi$ term from the original equation for the potential, since this term first produces terms of this order in the solution.

Amiet's solution is, however, fairly accurate over a much larger region of the k_1, M plane than (11) because some of the higher-order k_1 terms (k_1^3 and above) which contribute to the error in the latter series have been included in (15). In particular the series expansion of $S(\lambda)$ to order λ^2 is accurate over only a very small range of λ, and the error is therefore considerably reduced by retaining this factor to its full accuracy.

In those regions where the first-order corrections to Sears' solution implied by (15) do not occur, the correction is of second order as given by (10).

If Mugridge's incompressible lifting-line approximation (13) is expanded further for small k_1 and then the similarity rules are applied, the resulting series is the same to order ν_1^2 as (16), which was derived from Amiet's expansion, except for a difference in the second term of $-\nu_1^2/4i\lambda$.

Adamczyk (1974) has also derived an approximate formula for a similar problem which occurs when a swept aerofoil encounters a compressible sinusoidal gust. This formula is derived by the $O(\nu_1)$ approach in a similar way to those of Amiet and Mugridge and is therefore strictly a correct approximation only for small values of the frequency parameter. We have confined the analysis of the present paper to the lift coefficient on an aerofoil in an oblique compressible gust convected at the free-stream speed. It is of course quite possible, but more complicated, to extend the analysis to cover the pitching moment and other force coefficients, and the same approach can also be used to study other unsteady upwashes such as the Kemp (1952) type and those that arise from bending and torsional oscillations and incident acoustic waves. The general points concerning the order of correction terms made in this paper apply to all these, but we have concentrated here on the lift coefficient and the convected oblique gust because of their relevance to the study of the effects of turbulence on bending forces and noise generation.

In the case of an aerofoil in a turbulent compressible flow the analysis which we have given in this paper shows that the errors incurred by using (or alternatively the appropriate corrections to be made to) Sears' theory, i.e. an 'incompressible strip theory', are of order M^2c/β^2L, c^2/β^2L^2, M^2c/β^4L^2 and their associated logarithmic terms. Here L is the integral length scale of the turbulence, which is inversely proportional to the wavenumbers of the energy-containing eddies, and c is the chord of the aerofoil. Some of the approximate theories discussed above give the terms of order M^2c/β^2L correctly, but not all the other terms. By comparing the relative sizes of the three orders of magnitude above, approximate theories such as Amiet's should be expected to perform best if restricted to large scales of turbulence, but over a wide Mach number range.

6. Conclusions

General perturbation series have been derived to second order for the lift induced on an aerofoil by a compressible oblique sinusoidal gust. The results are given in (12) and (13). Comparison with Amiet's and other approximate solutions shows that these in general have incorrect asymptotic behaviour as the spanwise wavenumber k_2 or the compressible reduced frequency parameter k_1M/β^2 tends to zero, with k_1 held constant. On the other hand Amiet's and Mugridge's solutions do provide formulae which are approximately correct over the whole range of these parameters, and give the correct first-order terms in the series when k_1 tends to zero.

Filotas' solution provides a useful high wavenumber asymptote but seems to us to be less useful elsewhere since it involves a lengthy numerical summation.

Compressibility and three-dimensionality only introduce perturbations of second order into the basic two-dimensional solution for incompressible unsteady flow past an aerofoil, unless the frequency parameter k_1 also tends to zero, when the perturbation is of first order.

Part of this work was carried out while I. Kullar was supported by a Science Research Council Research Studentship.

REFERENCES

ADAMCZYK, J. J. 1971 Ph.D. thesis, University of Connecticut. (See also *United Aircraft Corp., East Hartford, Connecticut, Rep.* UAR-K97.)

ADAMCZYK, J. J. 1974 *N.A.S.A. Contractor Rep.* no. 2395.

AMIET, R. K. 1974 *A.I.A.A. J.* **12**, 252.

AMIET, R. K. 1975 *J. Sound Vib.* **41**, 407.

AMIET, R. K. 1976*a* *A.I.A.A. J.* **14**, 541.

AMIET, R. K. 1976*b* *J. Fluid Mech.* **75**, 545.

BISPLINGHOFF, R. L., ASHLEY, H. & HALFMAN, R. L. 1955 *Aeroelasticity.* Addison-Wesley.

FILOTAS, L. T. 1969 *N.A.S.A. Special Paper* no. **207**, p. 231.

GRAHAM, J. M. R. 1970*a* *Aero. Quart.* **21**, 182.

GRAHAM, J. M. R. 1970*b* *J. Fluid Mech.* **43**, 753.

JAMES, E. C. 1975 *J. Fluid Mech.* **70**, 753.

KEMP, N. H. 1952 *J. Aero. Sci.* **19**, 713.

KEMP, N. H. & HOMICZ, G. 1976 *A.I.A.A. J.* **14**, 1083.

MCKEOUGH, P. J. 1976 Ph.D. thesis, University of London.

MILES, J. W. 1950*a* *Quart. Appl. Math.* **8**, 350.

MILES, J. W. 1950*b* *J. Aero. Sci.* **17**, 181.
MUGRIDGE, B. D. 1971 *Aero. Quart.* **22**, 301.
OSBORNE, C. 1973 *A.I.A.A. J.* **11**, 205.
POSSIO, C. 1938 *L'Aerotechnica* **18**, 441.
REISSNER, E. 1951 *N.A.C.A. Tech. Note* no. 2363.
SEARS, W. R. 1941 *J. Aero. Sci.* **8**, 104.
VAN HOLTEN, TH. 1976 *J. Fluid Mech.* **77**, 561.